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Stochastic Processes and their Applications 129 (2019) 4556–4575

stochastic processes and their applications

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Tunneling behavior of Ising and Potts models in the low-temperature regime

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Received 31 August 2017; received in revised form 27 July 2018; accepted 3 December 2018
Available online 17 December 2018

Abstract

We consider the ferromagnetic q-state Potts model with zero external field in a finite volume and assume that its stochastic evolution is described by a Glauber-type dynamics parametrized by the inverse temperature β. Our analysis concerns the low-temperature regime $\beta \to \infty$, in which this multi-spin system has q stable equilibria. Focusing on grid graphs with various boundary conditions, we study the tunneling phenomena of the q-state Potts model, characterizing the asymptotic behavior of the first hitting times between stable equilibria as $\beta \to \infty$ in probability, in expectation, and in distribution and obtaining tight bounds on the mixing time as side-result.

Keywords: Ising model; Potts model; Glauber dynamics; Hitting times; Mixing time; Grid graphs

1. Introduction and main results

1.1. Model description

The Potts model is a canonical statistical physics model born as a natural extension [59] of the Ising model in which the number of possible local spins values/directions goes from two to a general integer number $q \in \mathbb{N}$.

The q-state Potts model is a spin system characterized by a set $S = \{1, \dots, q\}$ of spins values and by a finite graph $G = (V, E)$, which describes the spatial structure of the finite volume where the spins interact. A configuration $\sigma \in S^V$ assigns a spin value $\sigma(v) \in S$ to each vertex

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https://doi.org/10.1016/j.spa.2018.12.001

$v \in V$. The edge set E describes the pairs of vertices whose spins interact with each other. In the present paper we focus on *grid graphs*, i.e., finite two-dimensional rectangular lattices. More precisely, given two integers $K, L \geq 3$, we will take the graph G to be a $K \times L$ grid graph Λ with two possible boundary conditions: periodic and open.

Let $\mathcal{X} = S^V$ be the set of all possible spin configurations on the graph Λ. The *Hamiltonian* or *energy function* $H : \mathcal{X} \to \mathbb{R}$ associates an energy with each configuration $\sigma \in \mathcal{X}$ according to

$$H(\sigma) := -J_c \sum_{(v,w)\in E} \mathbb{1}_{\{\sigma(v)=\sigma(w)\}}, \quad \sigma \in \mathcal{X}, \tag{1}$$

where J_c is the *coupling or interaction constant*. Such an energy function corresponds to the situation where there is no external magnetic field and, in fact, $H(\sigma)$ describes only the local interactions between nearest-neighbor spins. The *Gibbs measure* for the q-state Potts model on Λ is the probability distribution on $\mathcal{X}$ defined by

$$\mu_\beta(\sigma) := \frac{e^{-\beta H(\sigma)}}{\sum_{\sigma'\in\mathcal{X}} e^{-\beta H(\sigma')}}, \quad \sigma \in \mathcal{X}, \tag{2}$$

where $\beta > 0$ is the *inverse temperature*. The Potts model is called *ferromagnetic* if $J_c > 0$ and *antiferromagnetic* if $J_c < 0$. In the ferromagnetic case, on which we focus in this paper, the Gibbs measure μ_β favors configurations where neighboring spins have the same value. On the contrary, in the antiferromagnetic case, neighboring spins are more likely not to be aligned. Without loss of generality, we take $J_c = 1$, since in absence of a magnetic field it amounts to rescaling of the temperature.

We assume the spin system evolves according to a Glauber-type dynamics described by a single-spin update Markov chain $\{X_t^\beta\}_{t\in\mathbb{N}}$ on $\mathcal{X}$ with transition probabilities between any pair of configurations $\sigma, \sigma' \in \mathcal{X}$ given by

$$P_\beta(\sigma, \sigma') := \begin{cases} Q(\sigma, \sigma')e^{-\beta[H(\sigma')-H(\sigma)]^+}, & \text{if } \sigma \neq \sigma', \\ 1 - \sum_{\eta\neq\sigma} P_\beta(\sigma, \eta), & \text{if } \sigma = \sigma', \end{cases} \tag{3}$$

where $Q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ is a *connectivity matrix* that allows only single-spin updates, namely $Q(\sigma, \sigma') := (q|V|)^{-1}\mathbb{1}_{\{|\{v\in V : \sigma(v)\neq\sigma'(v)\}|=1\}}$ for every $\sigma, \sigma' \in \mathcal{X}$. The matrix Q is clearly symmetric and irreducible, and the resulting dynamics P_β is reversible with respect to the Gibbs measure μ_β given in (2). One usually refers to the triplet $(\mathcal{X}, H, Q)$ as *energy landscape* and to (3) as *Metropolis transition probabilities*.

The considered Metropolis dynamics can be described in words as follows. At each step a vertex $v \in V$ and a spin value $k \in S$ are selected independently and uniformly at random and the current configuration $\sigma \in \mathcal{X}$ is updated in vertex v to spin k with a probability that depends only on the neighboring spins of v in view of (1).

1.2. Main results

In this work we focus on the analysis of the q-state ferromagnetic Potts model in the low-temperature regime $\beta \to \infty$, in which the system is in the so-called *ordered phase*, in which multiple equilibrium states coexist. Indeed, in this regime the stationary distribution μ_β concentrates around the global minima of the Hamiltonian H, which, since Λ is a connected graph, are exactly q and correspond to the configurations where all the spins have the same value. We denote them by $\mathbf{s}_1, \dots, \mathbf{s}_q$, with the convention that $\mathbf{s}_k \in \mathcal{X}$ is the configuration where all the spins are equal to k, i.e., $\mathbf{s}_k(v) = k$ for every $v \in V$. In the following we will refer to them as the *stable configurations* and denote their collection as $\mathcal{X}^s \subset \mathcal{X}$.

In the low-temperature regime these stable configurations and their basins of attraction become *traps*, in the sense that the Markov chain $\{X_t^\beta\}_{t\in\mathbb{N}}$ cannot move quickly between them. Intuitively, along any possible trajectory the Markov chain must visit mixed-spin configurations that are highly unlikely in view of (2) and the time to reach such configurations is correspondingly long. Due to these exponentially long transition times between stable configurations, the considered dynamics exhibits the so-called *slow* or *torpid mixing*.

We characterize the low-temperature behavior of the q-state Potts model in terms of both hitting times and mixing times. For a nonempty subset $A \subset \mathcal{X}$ and $\sigma \in \mathcal{X} \setminus A$, we denote by τ_A^σ the *first hitting time* of the subset A for the Markov chain $\{X_t^\beta\}_{t\in\mathbb{N}}$ with initial configuration σ at time $t = 0$. The hitting time τ_A^σ is often called *tunneling time* when both the starting and target configurations are stable configurations, i.e., $\{\sigma\} \cup A \subseteq \mathcal{X}^s$. For $\epsilon \in (0, 1)$ define the *mixing time* $t_\beta^{\mathrm{mix}}(\epsilon)$ of the Markov chain $\{X_t^\beta\}_{t\in\mathbb{N}}$ as $t_\beta^{\mathrm{mix}}(\epsilon) := \min\{n \geq 0 \,:\, \max_{x\in\mathcal{X}} \|P_\beta^n(x, \cdot) - \mu_\beta(\cdot)\|_{\mathrm{TV}} \leq \epsilon\}$, where $\|\nu - \nu'\|_{\mathrm{TV}}$ denotes the total variation distance between two probability distributions ν, ν' on $\mathcal{X}$. The mixing time $t_\beta^{\mathrm{mix}}(\epsilon)$ describes the rate of convergence of the Markov chain $\{X_t^\beta\}_{t\in\mathbb{N}}$ to its stationary distribution μ_β and is intimately related to the *spectral gap* ρ_β, defined in terms of the eigenvalues $1 = \lambda_\beta^{(1)} > \lambda_\beta^{(2)} \geq \cdots \geq \lambda_\beta^{(|\mathcal{X}|)} \geq -1$ of the transition matrix $(P_\beta(\sigma, \sigma'))_{\sigma,\sigma'\in\mathcal{X}}$ as $\rho_\beta := 1 - \lambda_\beta^{(2)}$.

The main result of this paper concerns the asymptotic behavior of the tunneling times between stable configurations: for any pair of stable configurations $\mathbf{s}, \mathbf{s}'$, we give asymptotic bounds in probability for $\tau^{\mathbf{s}}_{\mathcal{X}^s\setminus\{\mathbf{s}\}}$ and $\tau^{\mathbf{s}}_{\mathbf{s}'}$, identify the order of magnitude of their expected values and prove that their asymptotic rescaled distribution is exponential. We further identify the precise exponent at which the mixing time of the Markov chain $\{X_t^\beta\}_{t\in\mathbb{N}}$ asymptotically grows with β and show that it depends up to a constant factor on the smaller side length of Λ.

Theorem 1.1 (*Low-temperature Behavior of the Potts Model on Grid Graphs*)**.** *Consider the Metropolis Markov chain $\{X_t^\beta\}_{t\in\mathbb{N}}$ corresponding to the q-state Potts model on the $K \times L$ grid Λ with* $\max\{K, L\} \geq 3$. *Let $\Gamma(\Lambda) > 0$ be the constant defined as*

$$\Gamma(\Lambda) := \begin{cases} 2\min\{K, L\} + 2 & \text{if } \Lambda \text{ has periodic boundary conditions,} \\ \min\{K, L\} + 1 & \text{if } \Lambda \text{ has open boundary conditions.} \end{cases}$$

Then, for any $\mathbf{s}, \mathbf{s}' \in \mathcal{X}^s$, $\mathbf{s} \neq \mathbf{s}'$, the following statements hold:

(i) *For every* $\epsilon > 0$ $\lim_{\beta\to\infty} \mathbb{P}\Big(e^{\beta(\Gamma(\Lambda)-\epsilon)} < \tau^{\mathbf{s}}_{\mathcal{X}^s\setminus\{\mathbf{s}\}} \leq \tau^{\mathbf{s}}_{\mathbf{s}'} < e^{\beta(\Gamma(\Lambda)+\epsilon)}\Big) = 1$;

(ii) $\lim_{\beta\to\infty} \frac{1}{\beta} \log \mathbb{E}\tau^{\mathbf{s}}_{\mathbf{s}'} = \lim_{\beta\to\infty} \frac{1}{\beta} \log \mathbb{E}\tau^{\mathbf{s}}_{\mathcal{X}^s\setminus\{\mathbf{s}\}} = \Gamma(\Lambda)$;

(iii) $\frac{\tau^{\mathbf{s}}_{\mathcal{X}^s\setminus\{\mathbf{s}\}}}{\mathbb{E}\tau^{\mathbf{s}}_{\mathcal{X}^s\setminus\{\mathbf{s}\}}} \xrightarrow{d} \mathrm{Exp}(1), \quad$ *as* $\beta \to \infty$;

(iv) $\frac{\tau^{\mathbf{s}}_{\mathbf{s}'}}{\mathbb{E}\tau^{\mathbf{s}}_{\mathbf{s}'}} \xrightarrow{d} \mathrm{Exp}(1), \quad$ *as* $\beta \to \infty$;

(v) *For any* $\epsilon \in (0, 1)$ $\lim_{\beta\to\infty} \beta^{-1} \log t_\beta^{\mathrm{mix}}(\epsilon) = \Gamma(\Lambda)$ *and there exist two constants* $0 < c_1 \leq c_2 < \infty$ *independent of β such that* $c_1 e^{-\beta\Gamma(\Lambda)} \leq \rho_\beta \leq c_2 e^{-\beta\Gamma(\Lambda)}$ *for every* $\beta > 0$.

We remark that in the low-temperature limit the total number q of possible spin values does not appear in our main result because we focus on *logarithmic* equivalences and the number q does not affect the *order of magnitude* of the tunneling times and neither that of the mixing time. This is the case also for analogous results for mixing times of heat-bath and Swenden–Wang dynamics derived in [15], for which the dependence on the grid side length is the same. The bounds in [15] are valid for a more general d-dimensional grid, while ours are specialized for the case $d = 2$, for which we obtain sharper exponents.

In the degenerate case $K = L = 2$ not covered by Theorem 1.1, the same results still hold, but the correct exponents can be shown to be $\Gamma(\Lambda) = 4$ and $\Gamma(\Lambda) = 2$ for periodic and open boundary conditions, respectively. From our analysis it is easy to derive analogous results for a $K \times L$ grid graph Λ with semi-periodic boundary conditions (i.e., periodic on the horizontal boundaries and open on the vertical ones), for which the exponent $\Gamma(\Lambda)$ would be equal to $\min\{K + 2, 2L + 1\}$. We expect analogous results to hold also for rectangular regions Λ of other lattices (e.g. triangular, hexagonal, Kagome lattices) with an exponent $\Gamma(\Lambda)$ that would depend, up to a constant, on the minimum side length.

1.3. Related results and discussion

The Potts model is one of the most studied statistical physics models and is named after Renfrey Potts, who introduced the model in his Ph.D. thesis [59] in 1951. The model was related to the "clock model" or "planar Potts", a variant of which was introduced earlier in [4] and is known as the *Ashkin–Teller model*. The Potts model has been studied so extensively both by mathematicians and physicists that an exhaustive review of the related literature would be very long and out of the scope of this paper. Nevertheless, we now outline some related work that focus on the equilibrium or dynamical properties of the Potts model that are most relevant for this paper.

The equilibrium properties of the Potts model, such as the phase transition, critical temperature, and their dependence on q, have been studied on various infinite graphs, such as the square lattice $\mathbb{Z}^d$ [5,6], the triangular lattice [7,36], and the Bethe lattice [1,3,52]. If the underlying structure is described instead by a complete graph, then we obtain the mean-field version of Potts model, also known as *Curie–Weiss Potts model*, which received a lot of attention in the literature [29,34,35,41,64]. Another branch of research focuses more on the dynamical properties of the Potts model, investigating in particular mixing times for various types of dynamics, the most studied ones being Glauber [13,14,30,38–40,42,44,46,47], Swendsen–Wang dynamics [14,15,27,28,37,43–45,63]. The focus of this part of literature is to describe at a given temperature how the mixing time grows as a function the graph size $n = |V|$ and the number of colors q. In particular, the goal is to distinguish whether the considered dynamics has fast or slow mixing depending on the type of the graph and its properties, such as boundary conditions or dimensions in the specific case of grid graphs.

In the present paper we study the low-temperature behavior of the Potts model using the *pathwise approach* (see [58] for a systematic overview and further references) and its more recent extensions [25,53,56], but also other techniques have been successfully used in the literature to study tunneling and metastability phenomena, e.g. the *potential theoretical approach* (introduced in [16], for an overview see [17]) and the *martingale approach* [8–10].

Metastability is a dynamical property with a similar flavor as tunneling that has been studied for various spin systems. In particular, the metastability for the mean-field 3-state Potts model with a non-reversible dynamics has been studied in [51] for fixed temperature in the thermodynamic limit. In this paper we focus instead on the q-state Potts model on finite volume with Metropolis dynamics, for which we study the tunneling behavior. There is an extensive literature about the metastable behavior for the Ising model with small external magnetic field $h > 0$ on square lattices with Glauber dynamics, which relates our main result in the special case $q = 2$. More specifically, results have been derived for the finite-volume case in [9,11,19,23,49,50,57] and for the infinite-volume one in [18,20,31,61]. In the finite-volume case the energy barrier between $-\mathbf{1}$ and $+\mathbf{1}$ and, thus, the exponent $\Gamma(\Lambda)$ depend only the ratio J_c/h (which determines the length of the so-called *critical droplet*) and not on the grid sizes (which are always taken sufficiently large to contain the critical droplet).

Similar results for the hitting times of the Ising model with zero magnetic field have been proved in [62]. More precisely, the lower bound $\max_{\sigma\in\mathcal{X}} E\tau^{\sigma}_{S_\Lambda} \geq c(\beta)e^{\beta\alpha^* L^{d-1}}$ is derived for the hitting time of a certain subset S_Λ on the d-dimensional cube $\Lambda \subset \mathbb{Z}^d$ of side L, where α^* is a constant independent of Λ and β and $c(\beta) > 0$ is a function that does not depend on Λ. Since $+\mathbf{1} \in S_\Lambda$, our result can be seen as a refinement of [62, Proposition 2.3] in dimension $d = 2$, as (a) we identify the precise constant α^* showing how it depends on the type of boundary conditions, (b) we indirectly prove that $\lim_{\beta\to\infty} \frac{1}{\beta} \log c(\beta) = 0$, and (c) we derive a matching upper bound. Moreover, when $q = 2$ Theorem 1.1(iv) improves the estimates on the spectral gap of [62, Proposition 2.5], since we identify the exact exponent and obtain a matching upper bound for this quantity. In the special case of open boundary conditions, our result for the spectral gap should be compared with the estimates of [55, Theorem 4.1] (valid for more general dynamics) and the asymptotics for $L \to \infty$ proved in [21, Theorem 1.4]. For related equilibrium properties of the Ising model on finite lattices with zero magnetic field see also [22].

Results have been obtained for the metastability of the Ising model on the hypercube [48] and on certain types of random graphs [32,33]. The *Blume–Capel model* is another related 3-spin system which has been studied with similar techniques in [24,26,54]. Tunneling phenomena have been studied for other models with Metropolis dynamics: the *hard-core model* [56,66] and the *Widom–Rowlinson model* [67].

2. Geometry of Potts configurations and energy landscape analysis

This section is devoted to the analysis of some geometrical and combinatorial properties of the Potts configurations on grid graphs. This analysis will then be leveraged to prove some structural properties of the corresponding energy landscape $(\mathcal{X}, H, Q)$, which are presented in Theorem 2.1. These properties are precisely the model-dependent characteristics needed to exploit the general framework developed in [56] to derive our main result, Theorem 1.1, whose proof will be presented in Section 3.

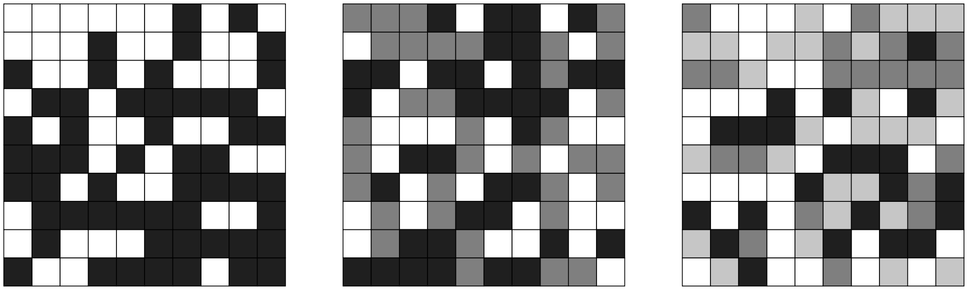
We first introduce some definition and notation that will be used in the rest of the paper. Since the connectivity matrix Q is irreducible, for each pair of configurations $\sigma, \sigma' \in \mathcal{X}$, $\sigma \neq \sigma'$, there exists a finite sequence ω of configurations $\omega_1, \dots, \omega_n \in \mathcal{X}$ such that $\omega_1 = \sigma$, $\omega_n = \sigma'$ and $Q(\omega_i, \omega_{i+1}) > 0$, for $i = 1, \dots, n-1$. We will refer to such a sequence as a *path* from σ to σ' and we will denote it by $\omega : \sigma \to \sigma'$.

Given a path $\omega = (\omega_1, \dots, \omega_n)$, we define its *height* as $\Phi_\omega := \max_{i=1,\dots,n} H(\omega_i)$. The *communication energy* between two configurations $\sigma, \sigma' \in \mathcal{X}$ is the minimum among the heights of all the paths from σ to σ', i.e., $\Phi(\sigma, \sigma') := \min_{\omega:\sigma\to\sigma'} \Phi_\omega = \min_{\omega:\sigma\to\sigma'} \max_{\eta\in\omega} H(\eta)$. Given two nonempty disjoint subsets $A, B \subset \mathcal{X}$, we define the communication energy between A and B by $\Phi(A, B) := \min_{\sigma\in A, \sigma'\in B} \Phi(\sigma, \sigma')$.

Theorem 2.1 (*Structural Properties of Energy Landscape*)**.** *Consider the energy landscape $(\mathcal{X}, H, Q)$ corresponding to the Potts model on a $K \times L$ grid Λ with* $\min\{K, L\} \geq 3$. *Then:*

(i) *For every $\mathbf{s}, \mathbf{s}' \in \mathcal{X}^s$, $\mathbf{s} \neq \mathbf{s}'$*

$$\Phi(\mathbf{s}, \mathbf{s}') - H(\mathbf{s}) = \Gamma(\Lambda) = \begin{cases} 2\min\{K, L\} + 2 & \text{if } \Lambda \text{ has periodic boundary conditions,} \\ \min\{K, L\} + 1 & \text{if } \Lambda \text{ has open boundary conditions.} \end{cases}$$



(a) Potts configuration with $q = 2$ (b) Potts configuration with $q = 3$ (c) Potts configuration with $q = 4$

Fig. 1. Examples of Potts configuration on the 10×10 grid.

(ii) *For every* $\sigma \in \mathcal{X} \setminus \mathcal{X}^s$ *the inequality* $\Phi(\sigma, \mathcal{X}^s) - H(\sigma) < \Gamma(\Lambda)$ *holds.*

The rest of the section is organized as follows: in Section 2.1 we introduce some useful notation and definitions that will be used throughout the section, while in Section 2.2 we describe the geometric properties of Potts configurations that will be of interest for our analysis. Later, in Section 2.3 we present an *expansion algorithm* for Potts configurations on grid graphs that will be leveraged to build paths of prescribed height between stable configurations. Section 2.4 is devoted to the derivation of lower bounds for the communication height between stable configurations and leads to Section 2.5, where we present the proof of Theorem 2.1.

2.1. Definitions and notation

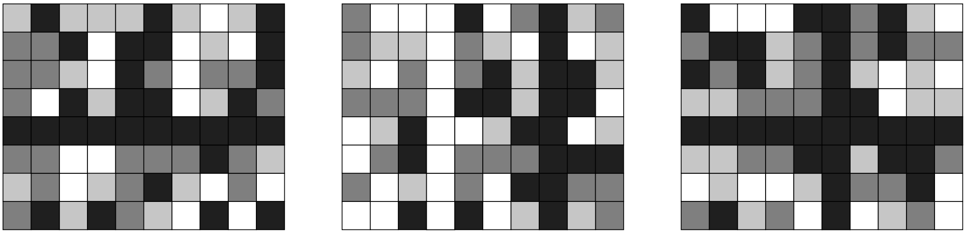
In this subsection we introduce some notation and definitions for the Potts model on a grid graph Λ, which are valid regardless of the chosen boundary conditions, unless specified otherwise.

A $K \times L$ grid graph $\Lambda = (V, E)$ has vertex set $V = \{0, \dots, L-1\} \times \{0, \dots, K-1\}$ and every vertex $v \in V$ is naturally identified by its coordinates (v_1, v_2), where v_1 denotes the column and v_2 the row where v lies. We denote by c_j, $j = 0, \dots, L-1$, the jth column of Λ, i.e., the collection of vertices whose horizontal coordinates are equal to j, and by r_i, $i = 0, \dots, K-1$, the ith row of Λ, i.e., the collection of vertices whose vertical coordinates are equal to i. It is convenient to visualize a q-state Potts configuration on a $K \times L$ grid graph Λ by coloring a $K \times L$ chessboard with q colors, one for each spin value, see three examples in Fig. 1.

Note that this representation respects the adjacency relations: the neighbors of a given vertex v are in one-to-one correspondence with the squares that share an edge with the square corresponding to v. In the rest of the paper, for brevity, we will interchangeably refer to the spin value of a vertex using its color.

Define the *energy gap* $\Delta H(\sigma)$ of a configuration $\sigma \in \mathcal{X}$ as the difference between its energy and the energy of any stable configuration, i.e., $\Delta H(\sigma) := H(\sigma) - H(\mathbf{s})$, for any $\mathbf{s} \in \mathcal{X}^s$. Given a configuration $\sigma \in \mathcal{X}$, we call an edge $e = (v, w) \in E$ *disagreeing* if it connects two vertices with different colors, i.e., $\sigma(v) \neq \sigma(w)$. From (1), it follows that $\Delta H(\sigma)$ is equal to the number of disagreeing edges that configuration σ has, since

$$\Delta H(\sigma) = H(\sigma) + |E| = |E| - \sum_{(v,w) \in E} \mathbb{1}_{\{\sigma(v) = \sigma(w)\}} = \sum_{(v,w) \in E} \mathbb{1}_{\{\sigma(v) \neq \sigma(w)\}}.$$



(a) A horizontal black bridge

(b) Two vertical bridges, one black and one white

(c) A black cross

Fig. 2. Example of configurations on a 8×10 grid graph displaying black bridges or a black cross.

The energy gap $\Delta H(\sigma)$ quantifies the *surface tension* between different spin clusters, since $\Delta H(\sigma)$ is equal to the total perimeter of the same-color clusters that configuration σ has. Indeed, the disagreeing edges of a Potts configuration σ on Λ are in one-to-one correspondence with the edges of the dual graph $\Lambda + \left(\frac{1}{2}, \frac{1}{2}\right)$ that are used to define the Peierls contour of the clusters.

The structure of the grid graph Λ allows us to rewrite the energy gap in a different form that will be crucial for our analysis. The edges of Λ can have either vertical or horizontal orientation, and we can partition the edge set E accordingly into two subsets of vertical edges E_v and horizontal edges E_h. Using this partition of E and (1), we can rewrite the energy gap $\Delta H(\sigma)$ as the sum of the contributions on horizontal and vertical edges, namely

$$\Delta H(\sigma) = \sum_{(v,w)\in E_v} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}} + \sum_{(v,w)\in E_h} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}}.$$

Let $\Delta H_{r_i}(\sigma)$ be the energy gap of a configuration $\sigma \in \mathcal{X}$ in the ith row, namely $\Delta H_{r_i}(\sigma) := \sum_{(v,w)\in r_i} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}}$, and $\Delta H_{c_j}(\sigma)$ that in the jth column, i.e., $\Delta H_{c_j}(\sigma) := \sum_{(v,w)\in c_j} \mathbb{1}_{\{\sigma(v)\neq\sigma(w)\}}$, where, with a minor abuse of notation, we write $(v, w) \in r_i$ (c_j, respectively) when $(v, w) \in E$ is a horizontal (vertical) edge that links two vertices v, w both on row r_i (column c_j). We can thus rewrite the energy gap of a configuration $\sigma \in \mathcal{X}$ as

$$\Delta H(\sigma) = \sum_{i=0}^{K-1} \Delta H_{r_i}(\sigma) + \sum_{j=0}^{L-1} \Delta H_{c_j}(\sigma). \tag{4}$$

Given $\sigma \in \mathcal{X}$ on Λ, a vertex $v \in V$, and $k \in \{1, \dots, q\}$, we define $\sigma^{v,k} \in \mathcal{X}$ to be the configuration obtained from σ by coloring the vertex v with color k, i.e., $\sigma^{v,k}(w) = k$ if $w = v$ and $\sigma^{v,k}(w) = \sigma(w)$ otherwise.

2.2. Local geometric properties: Bridges and crosses

In this subsection we will introduce some geometric features of Potts configurations on a $K \times L$ grid graph Λ and study how they are related with their corresponding energy.

We say that a configuration $\sigma \in \mathcal{X}$ has a *horizontal bridge* on a row if all the vertices on that row have the same color. *Vertical bridges* are defined analogously. A few examples of bridges are illustrated in Fig. 2(a) and (b). An immediate consequence of the structure of rows and columns of the grid graph Λ is that a Potts configuration cannot display simultaneously a horizontal and a vertical bridge of different colors. Hence, if a configuration $\sigma \in \mathcal{X}$ has both a vertical and a horizontal bridge, they must be of the same color and we refer to them as a *cross*, see an example in Fig. 2(c). If the specific color $k \in \{1, \dots, q\}$ of bridges (crosses) is relevant, we will refer to them as *k-bridges* (*k-crosses*) or specify their color.

Lemma 2.2 (*Bridges and Zero Energy Gap Rows/Columns*)**.** *The following properties hold for every Potts configuration $\sigma \in \mathcal{X}$ on a grid graph Λ:*

(a) *$\Delta H_r(\sigma) = 0$ if and only if σ has a horizontal bridge on row r;*
(b) *$\Delta H_c(\sigma) = 0$ if and only if σ has a vertical bridge on column c.*

Furthermore, if Λ has periodic boundary conditions, then

(c) *If σ has no horizontal bridge on row r, then $\Delta H_r(\sigma) \geq 2$;*
(d) *If σ has no vertical bridge on column c, then $\Delta H_c(\sigma) \geq 2$.*

2.3. Expansion algorithm and reference path

In this subsection we introduce the *expansion algorithm*, a procedure that can be used to create a path to one of the stable configurations consisting of single-site updates allowed by the dynamics. The algorithm is presented in Proposition 2.3 and will be used twice: first to construct a path between any pair of stable configurations with a prescribed height (Proposition 2.4) and later to show that every Potts configuration on Λ can be reduced to a stable configuration with a maximum energy gap strictly smaller than $\Gamma(\Lambda)$, proving Theorem 2.1(ii).

Not every configuration $\sigma \in \mathcal{X}$ is a suitable starting point for the expansion algorithm: indeed, we require that there exists a monochromatic bridge in σ. The procedure gradually "expands" this monochromatic bridge by progressively changing the color of the vertices in the adjacent columns until the corresponding stable configuration is obtained: this is the reason for the name *expansion algorithm*. We remark that the fact that our algorithm makes a cluster grow gradually column by column (or row by row) is not crucial, and in fact we could have defined a more general procedure leveraging the vertex-isoperimetric order for grid graphs, known both for periodic and open boundary conditions [2,12,60,65]. We choose to present here a procedure based on the row and column structure of Λ as it is more intuitive and eventually yields the same energy bounds. The following proposition summarizes our findings for both types of boundary conditions.

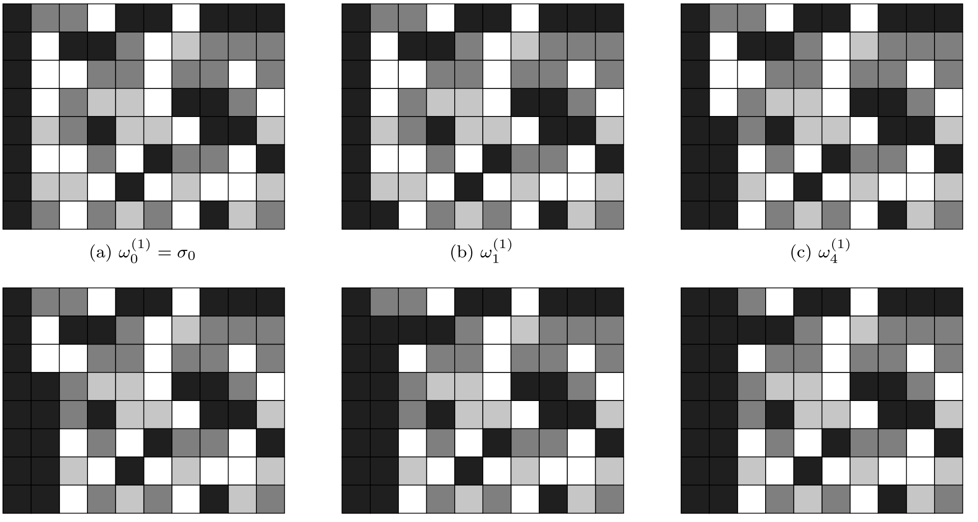
Proposition 2.3 (*Expansion Algorithm for Grid Graphs*)**.** *Let $\sigma \in \mathcal{X}$ be a Potts configuration on a grid graph Λ. If σ has a monochromatic k-bridge, then there exists a path $\omega : \sigma \to \mathbf{s}_k$ such that*

$$\Phi_\omega - H(\sigma) \leq \begin{cases} 2 & \text{if } \Lambda \text{ has periodic boundary conditions,} \\ 1 & \text{if } \Lambda \text{ has open boundary conditions.} \end{cases}$$

Proof. Consider first the case of periodic boundary conditions. We can assume that the k-bridge that σ has is vertical and lies on the first column c_0, modulo columns relabeling. If the k-bridge was horizontal, the argument still works simply interchanging the role of rows and columns. In the rest of the proof we associate the color black to the spin value k. We now describe an iterative procedure that builds a path $\omega : \sigma \to \mathbf{s}_k$ as concatenation of L paths $\omega^{(1)}, \dots, \omega^{(L)}$ and using the following *intermediate configurations* σ_i, for every $i = 0, \dots, L$:

$$\sigma_i(v) := \begin{cases} k & \text{if } v \in \bigcup_{j=0}^{i} c_j, \\ \sigma(v) & \text{if } v \in V \setminus \bigcup_{j=0}^{i} c_j. \end{cases}$$

For every $i = 1, \dots, L$ we will define a path $\omega^{(i)} : \sigma_{i-1} \to \sigma_i$ of length K so that along path $\omega^{(i)}$, $i = 1, \dots, L$, the vertices on ith column are progressively colored in black. Fig. 3



(a) $\omega_0^{(1)} = \sigma_0$ (b) $\omega_1^{(1)}$ (c) $\omega_4^{(1)}$

(d) $\omega_5^{(1)}$ (e) $\omega_7^{(1)}$ (f) $\omega_8^{(1)} = \sigma_1$

Fig. 3. Illustration of some configurations along the path $\omega^{(1)} : \sigma_0 \to \sigma_1$ on a 8×10 grid.

depicts some configurations along the path $\omega^{(1)}$. Set $\omega_0^{(i)} = \sigma_{i-1}$ and for any $m = 1, \dots, K$ define the configuration $\omega_m^{(i)}$ from the previous one by coloring as black the vertex $(i, m-1)$, i.e., $\omega_m^{(i)} := (\omega_{m-1}^{(i)})^{(i,m-1),k}$, for $m = 1, \dots, K$.

The energy cost of these single-vertex updates satisfies the following inequalities:

$$H(\omega_m^{(i)}) - H(\omega_{m-1}^{(i)}) \leq \begin{cases} 2 & \text{if } m = 1, \\ 0 & \text{if } 1 < m < K, \\ -2 & \text{if } m = K. \end{cases} \tag{5}$$

By updating a configuration only in a given vertex v, the edges that can change from agreeing to disagreeing and vice-versa are only those incident to v. Given $\eta \in \mathcal{X}$ and $v \in V$, denote by $d_v(\eta) = \sum_{w \in V : (v,w) \in E} \mathbb{1}_{\{\eta(v) \neq \eta(w)\}}$ the number of disagreeing edges incident to vertex v in configuration η, so that

$$H(\omega_m^{(i)}) - H(\omega_{m-1}^{(i)}) = d_{(i,m-1)}(\omega_m^{(i)}) - d_{(i,m-1)}(\omega_{m-1}^{(i)}). \tag{6}$$

Denote $v = (i, m-1)$. If v is already black, the step is void and trivially $H(\omega_m^{(i)}) - H(\omega_{m-1}^{(i)}) = 0$. If v is not black, using identity (6), inequality (5) is proved distinguishing three different cases, which are illustrated below in Fig. 3, respectively in (a)–(b) for $m = 1$, (c)–(d) for $1 < m < K$, and (e)–(f) for $m = K$.

- If $m = 1$, then $d_v(\omega_{m-1}^{(i)}) \geq 1$, since v is not black and, as such, disagrees at least with its left neighbor (black by construction), while $d_v(\omega_m^{(i)}) \leq 3$, since $d_v(\omega_m^{(i)}) \neq 4$ as at least the left neighbor of v is also black.
- If $1 < m < K$, then $d_v(\omega_{m-1}^{(i)}) \geq 2$, since v is not black and, as such, disagrees at least with its left and bottom neighbor (both black by construction), while $d_v(\omega_m^{(i)}) \leq 2$, since v is black and agrees at least with its left and bottom neighbor (both black by construction).
- If $m = K$, then $d_v(\omega_{m-1}^{(i)}) \geq 3$, since v is not black and as such it disagrees at least with its left, top, and bottom neighbors (that are black by construction), while $d_v(\omega_m^{(i)}) \leq 1$, since

v is black and agrees at least with its left, top, and bottom neighbors (all three black by construction).

The inequalities in (5) imply that $\Phi_{\omega^{(i)}} - H(\sigma_{i-1}) \leq 2$ for every $i = 1, \dots, L-1$. Therefore, by concatenating all the paths $\omega^{(1)}, \dots, \omega^{(L-1)}$ we obtain a path $\omega : \sigma \to \mathbf{s}_k$ such that $\Phi_\omega - H(\sigma) \leq 2$.

As far as the case of open boundary conditions is concerned, instead of giving a full description of the expansion algorithm, we will only briefly outline the main differences with respect to the one we just presented. Let $\sigma \in \mathcal{X}$ be the configuration displaying a black bridge, which, as before, we can assume to be vertical. There are two tweaks necessaries to adapt the algorithm described earlier to this scenario:

1. Let c^* be the column where the bridge lies. The columns of a grid graph with open boundary conditions are not identical and, therefore, we cannot assume without loss of generality that c^* is the first column c_0. The procedure described previously can be used to expand the bridge first to the right of c^*, until the open boundary of Λ is reached, and then analogously to the left of c^* ("mirroring" the moves described earlier).
2. Every new column is started by updating its bottom-most vertex (which in view of the open boundaries has at most 3 neighbors) and is completed by updating the topmost vertex (also having at most 3 neighbors). By revisiting the previous calculations, we can control the energy differences along any path $\omega^{(i)}$, namely
$$H(\omega^{(i)}_m) - H(\omega^{(i)}_{m-1}) \leq \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } 1 < m < K, \\ -1 & \text{if } m = K. \end{cases}$$
Therefore, $H(\sigma_i) \leq H(\sigma_{i-1})$ and $\Phi_{\omega^{(i)}} - H(\sigma_{i-1}) \leq 1$ and the path ω obtained by concatenating $\omega^{(1)}, \dots, \omega^{(L)}$ then satisfies the inequality $\Phi_\omega - H(\sigma) \leq 1$. □

Using the expansion algorithm we build a path between any pair of stable configurations with a prescribed height, to which we will refer as *reference path*.

Proposition 2.4 (*Reference Path*). *Consider the Potts model on a* $K \times L$ *grid* Λ *with* $\min\{K, L\} \geq 3$. *For every pair of stable configurations* $\mathbf{s}, \mathbf{s}' \in \mathcal{X}^s$, $\mathbf{s} \neq \mathbf{s}'$, *there exists a reference path* $\omega^* : \mathbf{s} \to \mathbf{s}'$ *such that*
$$\Phi_{\omega^*} - H(\mathbf{s}) = \begin{cases} 2\min\{K, L\} + 2 & \textit{if } \Lambda \textit{ has periodic boundary conditions,} \\ \min\{K, L\} + 1 & \textit{if } \Lambda \textit{ has open boundary conditions.} \end{cases}$$

Proof. Consider first the case of periodic boundary conditions and assume $K \leq L$. The construction of the reference path is analogous when $K > L$ and is obtained simply by interchanging the role of rows and columns.

Let σ^* be the configuration that agrees with the target configuration $\mathbf{s}'$ on the first column c_0 and elsewhere with the starting configuration $\mathbf{s}$. We construct a reference path ω^* from $\mathbf{s}$ to $\mathbf{s}'$ such that $\Phi_{\omega^*} - H(\mathbf{s}) = 2K + 2$ as the concatenation of two paths, $\omega^{(1)} : \mathbf{s} \to \sigma^*$ and $\omega^{(2)} : \sigma^* \to \mathbf{s}'$, such that $\Phi_{\omega^{(1)}} = H(\mathbf{s}) + 2K$ and $\Phi_{\omega^{(2)}} = H(\mathbf{s}) + 2K + 2$. For simplicity we color to the vertices whose spins agree with $\mathbf{s}$ as white and the one agreeing with $\mathbf{s}'$ as black, see Fig. 4 for some snapshots of the reference path ω.

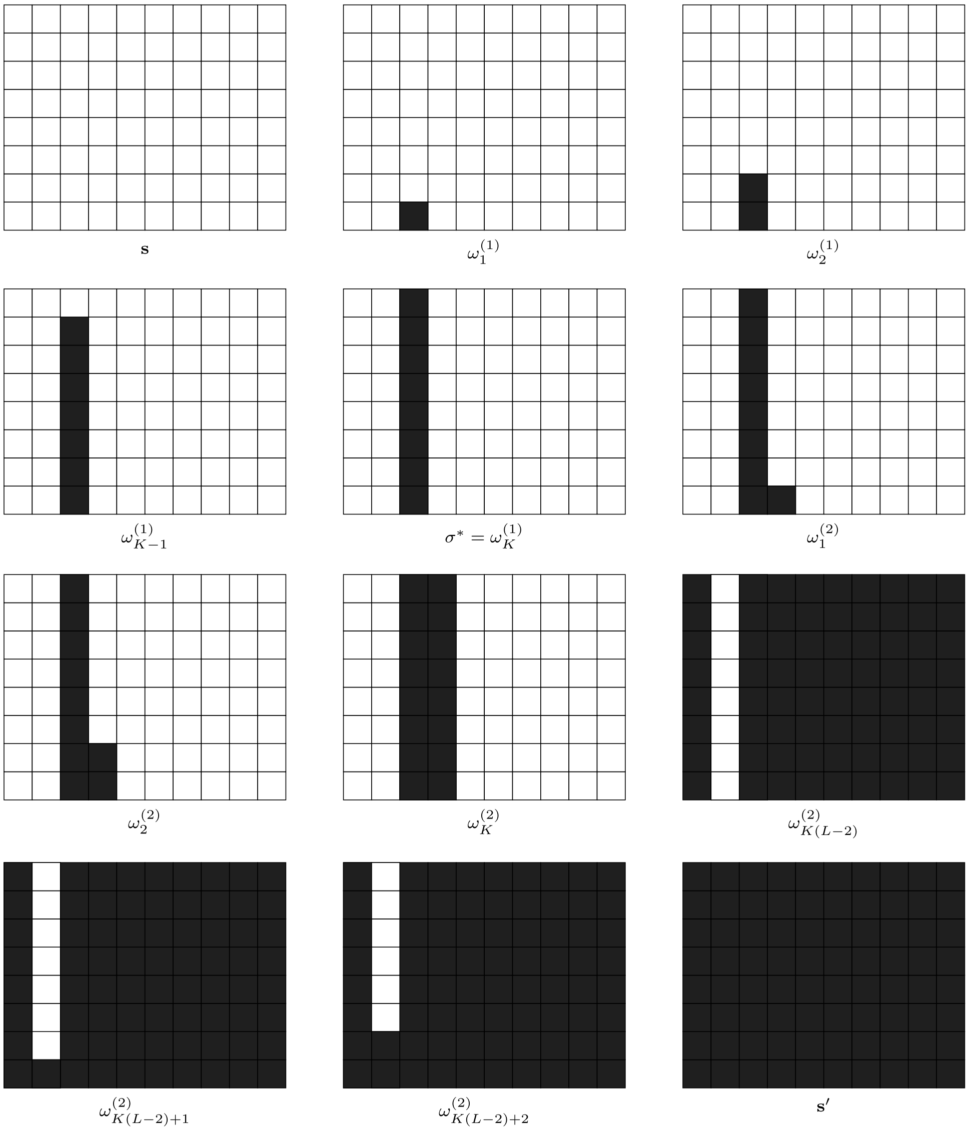


Fig. 4. Reference path in the case $K \le L$ (column c_0 is shifted to the right to visualize better the effect of the boundary).

The path $\omega^{(1)}$ is the path $(\omega_0^{(1)}, \dots, \omega_K^{(1)})$ of length K starting from $\omega_0^{(1)} = \mathbf{s}$ and obtained iteratively by coloring at step i vertex $(0, i-1)$ as black. It is easy to check that

$$H(\omega_i^{(1)}) - H(\omega_{i-1}^{(1)}) = \begin{cases} 4 & \text{if } i = 1, \\ 2 & \text{if } i = 2, \dots, K-1, \\ 0 & \text{if } i = K. \end{cases}$$

The configurations with the highest energy along $\omega^{(1)}$ are $\omega_{K-1}^{(1)}$ and $\omega_K^{(1)} = \sigma^*$, as $\Delta H(\omega_{K-1}^{(1)}) = 2K = \Delta H(\sigma^*)$, and therefore $\Phi_{\omega^{(1)}} = H(\sigma^*) = H(\mathbf{s}) + 2K$. The newly obtained configuration

σ^* has a monochromatic black bridge on c_0 and as such is a suitable starting configuration for the expansion algorithm introduced earlier. In view of Proposition 2.3, such an algorithm outputs a path $\omega^{(2)} : \sigma^* \to \mathbf{s}'$ such that $\Phi_{\omega^{(2)}} = H(\sigma^*) + 2 = H(\mathbf{s}) + 2K + 2$.

In the case where Λ has open boundary conditions, the exact same reference path yields the desired identity. By reviewing the calculations for the maximum energy along the paths $\omega^{(1)}$ and $\omega^{(2)}$ it is easy to show that $\Phi_{\omega^{(1)}} = H(\mathbf{s}) + K$ and $\Phi_{\omega^{(2)}} = H(\mathbf{s}) + K + 1$, from which the conclusion readily follows. $\square$

2.4. Communication energy between stable configurations

Given a configuration $\sigma \in \mathcal{X}$ and a spin value $k \in \{1, \dots, q\}$, let $B_k(\sigma) \in \mathbb{N} \cup \{0\}$ be the total number of k-bridges (both horizontal and vertical) that configuration σ has. The next lemma shows how this quantity evolves with single-spin updates and relates its increments with geometric properties of the spin configurations.

Lemma 2.5 (*Bridges Creation and Deletion*)**.** *Let $\sigma, \sigma' \in \mathcal{X}$ be two Potts configuration that differ by a single-spin update, i.e., such that $Q(\sigma, \sigma') > 0$. Then, for every $k \in \{1, \dots, q\}$ we have that $B_k(\sigma') - B_k(\sigma) \in \{-2, -1, 0, 1, 2\}$, and $B_k(\sigma') - B_k(\sigma) = 2$ if and only if σ' a k-cross that σ does not have.*

This lemma, whose easy proof is omitted, states that at most two bridges of a given color can be created or destroyed by a single-spin update and that, if two bridges are created simultaneously, they must be orthogonal (one horizontal and one vertical). The total number of k-bridges is the key quantity to exactly characterize the minimum height of any path between stable configurations, as illustrated by the next proposition.

Proposition 2.6 (*Communication Energy Lower Bound*)**.** *Consider the Potts model on a $K \times L$ grid with $\max\{K, L\} \geq 3$. Then, for every $\mathbf{s}, \mathbf{s}' \in \mathcal{X}^s$, with $\mathbf{s} \neq \mathbf{s}'$, the following inequality holds*

$$\Phi(\mathbf{s}, \mathbf{s}') - H(\mathbf{s}) \geq \begin{cases} 2\min\{K, L\} + 2 & \text{if } \Lambda \text{ has periodic boundary conditions,} \\ \min\{K, L\} + 1 & \text{if } \Lambda \text{ has open boundary conditions.} \end{cases}$$

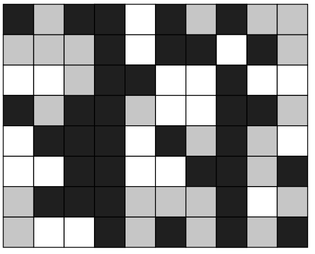
Proof. Consider first the case of periodic boundary conditions. It is enough to show that along every path $\omega : \mathbf{s} \to \mathbf{s}'$ in $\mathcal{X}$ there exists at least one configuration with energy gap not smaller than $2\min\{K, L\} + 2$. Let $k \in \{1, \dots, q\}$ be the spin value such that $\mathbf{s}' = \mathbf{s}_k$. In the rest of the proof we will associate the color black to the spin value k and, in particular, we will refer to k-bridges and k-crosses as black bridges and crosses, respectively.

Consider a path ω from $\mathbf{s}$ to $\mathbf{s}'$ of length n. Note that $\mathbf{s}$ has no black bridges, i.e., $B_k(\mathbf{s}) = 0$, while $\mathbf{s}'$ is has $B_k(\mathbf{s}') = K + L$ black bridges. Hence, there exists a configuration along the path ω that is the first to have at least two black bridges and let $m^* := \min\{m \leq n \mid B_k(\omega_m) \geq 2\} \in \mathbb{N}$ be the corresponding index. We claim that the total energy gap of the configuration ω_{m^*-1} satisfies the following inequality

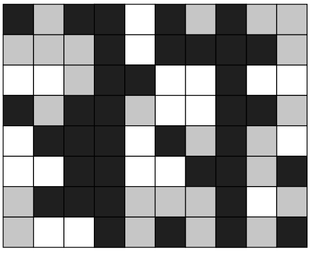
$$\Delta H(\omega_{m^*-1}) \geq 2\min\{K, L\} + 2.$$

We prove this claim by considering separately three scenarios:

(a) ω_{m^*} displays only vertical black bridges;
(b) ω_{m^*} displays only horizontal black bridges;
(c) ω_{m^*} displays at least one black cross.



(a) One black bridge on column c



(b) Two bridges on columns c and c'

Fig. 5. Examples of configuration ω_{m^*} in scenario (a).

(a) By definition of m^*, we have $B_k(\omega_{m^*-1}) \leq 1$ and $B_k(\omega_{m^*}) \geq 2$. Moreover, $B_k(\omega_{m^*}) - B_k(\omega_{m^*-1}) < 2$, since otherwise ω_{m^*} would have a black cross in view of Lemma 2.5. Hence, $B_k(\omega_{m^*-1}) = 1$ and $B_k(\omega_{m^*}) = 2$ and configuration ω_{m^*} has exactly two vertical black bridges, say on columns c and c', see an example in Fig. 5. Since ω_{m^*-1} and ω_{m^*} differ by a single-spin update and $B_k(\omega_{m^*-1}) = 1$, it follows that configuration ω_{m^*-1} has only one vertical k-bridge, say on column c, and all black vertices but one on column c'. In particular, ω_{m^*-1} has no vertical bridge on column c' and, therefore, by Lemma 2.2(d),

$$\Delta H_{c'}(\omega_{m^*-1}) \geq 2. \tag{7}$$

We claim that ω_{m^*-1} cannot have any horizontal bridge. Indeed the presence of a black horizontal bridge in some row would imply that $B_k(\omega_{m^*-1}) \geq 2$ (since ω_{m^*-1} has by construction at least a vertical black bridge on column c), contradicting the definition of m^*; furthermore, there cannot be non-black horizontal bridges either due to the presence of the black bridge in column c. By Lemma 2.2(c), the absence of horizontal bridges yields that $\Delta H_r(\omega_{m^*-1}) \geq 2$ for every row r and thus

$$\sum_{i=0}^{K-1} \Delta H_{r_i}(\omega_{m^*-1}) \geq 2K. \tag{8}$$

Using identity (4), inequalities (7) and (8) yield $\Delta H(\omega_{m^*-1}) \geq \Delta H_{c'}(\omega_{m^*-1}) + \sum_{i=0}^{K-1} \Delta H_{r_i}(\omega_{m^*-1}) \geq 2K + 2$.

(b) Arguing like in (a) but interchanging the role of rows and columns, one can show that $\Delta H(\omega_{m^*-1}) \geq 2L + 2$.

(c) Assume now that ω_{m^*} displays at least one black cross. By definition of m^*, the quantity $B_k(\omega_{m^*-1})$ can take only two values, 0 or 1, and we consider these two cases separately.

Assume first that $B_k(\omega_{m^*-1}) = 0$, which means that ω_{m^*-1} has no black bridges, see an example in Fig. 6.

Since ω_{m^*-1} and ω_{m^*} differ by a single-spin update, Lemma 2.5 gives that $B_k(\omega_{m^*}) \leq 2$ and thus we can conclude that $B_k(\omega_{m^*}) = 2$, by definition of m^*. Lemma 2.5 implies further that ω_{m^*} displays a unique black cross and denote by $\hat{r}$ and $\hat{c}$ the row and column on which it lies. Since $B_k(\omega_{m^*-1}) = 0$, the horizontal and vertical black bridges that ω_{m^*} has must have been created simultaneously by updating the spin in the vertex $\hat{v} = \hat{r} \cap \hat{c}$. Hence, by construction, $\omega_{m^*-1}(v) = k$ for every $v \in \hat{r} \cup \hat{c}$, $v \neq \hat{v}$, which means that there is a black vertex in every row and column. Consequently, configuration ω_{m^*-1} cannot have any non-black (horizontal or vertical) bridges, and, since we assumed $B_k(\omega_{m^*-1}) = 0$, we conclude that ω_{m^*-1} has *no* bridges

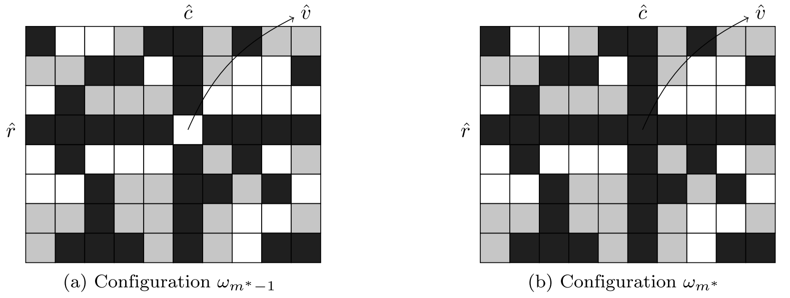


Fig. 6. Example for scenario (c) of a configuration ω_{m^*-1} such that $B_k(\omega_{m^*-1}) = 0$.

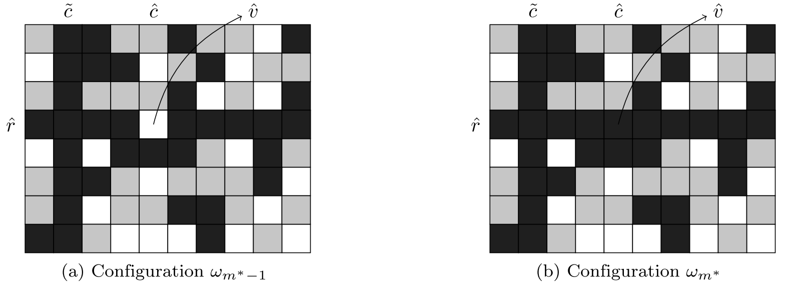


Fig. 7. Example for scenario (c) of a configuration ω_{m^*-1} such that $B_k(\omega_{m^*-1}) = 1$.

of *any* color. Therefore, by Lemma 2.2(c)–(d), the energy gap is not smaller than 2 in every row and column, and, hence,

$$\sum_{i=0}^{K-1} \Delta H_{r_i}(\omega_{m^*-1}) \geq 2K \quad \text{and} \quad \sum_{j=0}^{L-1} \Delta H_{c_j}(\omega_{m^*-1}) \geq 2L.$$

and thus, using identity (4), we obtain $\Delta H(\omega_{m^*-1}) \geq 2K + 2L \geq 2\min\{K, L\} + 2\max\{K, L\} > 2\min\{K, L\} + 2$.

Consider now the scenario in which $B_k(\omega_{m^*-1}) = 1$, in which ω_{m^*-1} has a unique black bridge, see Fig. 7. Assume such black bridge is vertical and lies in column $\tilde{c}$ (if it is horizontal, the proof is identical after interchanging the role of rows and columns). Its presence makes the existence of any horizontal non-black bridge impossible in ω_{m^*-1}. Furthermore, by assumption ω_{m^*-1} has no horizontal black bridges and Lemma 2.2(c) then yields

$$\sum_{i=0}^{K-1} \Delta H_{r_i}(\omega_{m^*-1}) \geq 2K. \tag{9}$$

Since ω_{m^*-1} and ω_{m^*} differ by a single-spin update, the presence of a black cross ω_{m^*} and the absence of horizontal black bridges in ω_{m^*-1} imply that ω_{m^*} has a unique horizontal black bridge, say on row $\hat{r}$. By construction, the vertex, say $\hat{v}$, where ω_{m^*} and ω_{m^*-1} differ must lie in such a row, $\omega_{m^*-1}(\hat{v}) \neq k$, and $\omega_{m^*-1}(v) = k$ for every $v \in \hat{r}, v \neq \hat{v}$. Let $\hat{c}$ be the column where $\hat{v}$ lies. The black vertices in row $\hat{r}$ implies that configuration ω_{m^*-1} has no vertical l-bridge with $l \neq k$ in every column $c \neq \hat{c}, \tilde{c}$. Lemma 2.2(d) then yields that in each of these $L - 2$ columns the energy gap is greater than or equal to 2 and, thus,

$$\sum_{j=0}^{L-1} \Delta H_{c_j}(\omega_{m^*-1}) \geq 2(L-2) = 2L-4. \tag{10}$$

Inequalities (9) and (10) combined yield $\Delta H(\omega_{m^*-1}) \geq 2K + 2L - 4 \geq 2\min\{K, L\} + 2\max\{K, L\} - 4 \geq 2\min\{K, L\} + 2$, where the last inequality holds since $\max\{K, L\} \geq 3$.

The proof in the case of open boundary conditions is very similar and thus omitted. The only minor tweak necessary is due to the fact that the lower bound for the energy gap on rows or columns without bridges is different due to the open boundary conditions. As illustrated in Lemma 2.2, a row or column without bridges has energy gap not smaller than 2 when Λ has periodic boundary conditions, while we only know that is non-zero (and in particular greater than or equal to 1) when Λ has open boundary conditions. By adjusting this factor in all the inequalities derived above, one gets the desired lower bound for the communication energy $\Phi(\mathbf{s}, \mathbf{s}')$. $\square$

2.5. *Proof of Theorem 2.1*

(i) The proof readily follows by combining the upper bound for $\Phi(\mathbf{s}, \mathbf{s}')$ given by the reference path constructed in Proposition 2.4 with the matching lower bound obtained in Proposition 2.6.

(ii) We first consider the case of (a) periodic boundary conditions and later that of (b) open boundary conditions. We henceforth assume $K \leq L$, since one can argue similarly interchanging rows and columns when $K > L$.

(a) Consider a configuration $\sigma \in \mathcal{X} \setminus \mathcal{X}^s$. If σ has a (vertical or horizontal) k-bridge for some $k = 1, \dots, q$, then the expansion algorithm can be used to build a path $\omega : \sigma \to \mathbf{s}_k$ such that $\Phi_\omega \leq H(\sigma) + 2$ and the proof is concluded. On the other hand, if σ has no (vertical or horizontal) bridges, consider the column c^* with the largest number of vertices of the same color, say black, and let k be the associated spin value. Define

$$\sigma^*(v) := \begin{cases} \sigma(v) & \text{if } v \in V \setminus c^*, \\ k & \text{if } v \in c^*. \end{cases}$$

Let $m := |\{v \in V : \sigma(v) \neq \sigma^*(v)\}|$ be the number of vertices in which configurations σ and σ^* differ, which is precisely the number of non-black vertices that configuration σ has on column c^*, since $\{v \in V : \sigma(v) \neq \sigma^*(v)\} = \{v \in c^* : \sigma(v) \neq k\}$. In particular, by construction, $m < K = |c^*|$. We build a path $\omega^{(1)} : \sigma \to \sigma^*$ along which these m non-black vertices are progressively colored as black. The order in which these vertices are updated is crucial to obtain the desired bound for Φ_ω: more specifically for every step $i = 1, \dots, m$, consider a vertex $v_i \in c^*$ such that $\omega^{(1)}_{i-1}(v) \neq k$ and with at least one black neighbors on column c^* and then obtain the new configuration $\omega^{(1)}_i$ from $\omega^{(1)}_{i-1}$ by coloring such vertex as black.

The way in which the vertices $v_1, \dots, v_m$ are progressively chosen guarantees that $\Delta H(\omega^{(1)}_i) \leq \Delta H(\omega^{(1)}_{i-1}) + 2$, for every $i = 1, \dots, m-1$, since at most two disagreements are created by coloring vertex v_i as black, and that $\Delta H(\omega^{(1)}_m) \leq \Delta H(\omega^{(1)}_{m-1})$, since vertex v_m has by construction exactly two black neighbors on column c^*. Hence, the path $\omega^{(1)}$ is such that $\Phi_{\omega^{(1)}} - H(\sigma) \leq 2(m-1)$. Configuration σ^* has a vertical black bridge and thus the expansion algorithm yields a path $\omega^{(2)} : \sigma^* \to \mathbf{s}_k$ such that $\Phi_{\omega^{(2)}} - H(\sigma^*) \leq 2$. The concatenation of $\omega^{(1)}$ and $\omega^{(2)}$ then is a path from σ to $\mathbf{s}_k$ that guarantees that $\Phi(\sigma, \mathcal{X}^s) - H(\sigma) \leq 2(m-1) + 2 \leq 2m < 2K < 2K + 2$.

(b) In the case of open boundary conditions we only briefly need to review the calculations done in (a). If σ has a bridge on the first column, then the expansion algorithm guarantees that $\Phi(\sigma, \mathcal{X}^s) - H(\sigma) \leq 1$. If there is no bridge there, let k be the most present spin value on the first column and associate the color black with it. Define the configuration σ^* obtained from σ by coloring as black all vertices on the first column, i.e., $\sigma^*(v) = k$ if $v \in c_0$ and $\sigma^*(v) = \sigma(v)$ otherwise. Similarly to case (a), we define a path from σ to $\mathcal{X}^s$ using σ^* as intermediate configuration. If m denotes the number of vertices in which configurations σ and σ^* differ, we have $m < K$ since we assumed that σ has no bridge on the first column. By progressively coloring them as black, always updating a vertex with at least one black neighboring vertex on c_0, the energy cost is no larger than 1 for every vertex newly colored in black (thanks to the open boundary conditions). In particular, coloring the last non-black vertex on column c_0 costs 0 or less, since by construction it had at most one disagreeing neighbor. The path we defined in this way implies that $\Phi(\sigma, \sigma^*) - H(\sigma) \leq m - 1$. We can then concatenate to this path another one built using the expansion algorithm (applicable because σ^* has a black bridge), obtaining in this way a path from σ to $\mathbf{s}_k \in \mathcal{X}^s$ and looking at its height we conclude that $\Phi(\sigma, \mathcal{X}^s) - H(\sigma) \leq (m-1) + 1 \leq m < K < K + 1$. □

3. Proof of Theorem 1.1

(i)–(ii) Consider the target stable configuration $\mathbf{s}' \in \mathcal{X}^s$. We first claim that

$$\forall \sigma \neq \mathbf{s}' \qquad \Phi(\sigma, \mathbf{s}') - H(\sigma) \leq \Gamma(\Lambda). \tag{11}$$

Assuming this inequality holds, then the energy barrier between stable configurations is the largest across the whole energy landscape. The proofs of statements (i) and (ii) readily follow by applying [56, Corollary 3.16] and [56, Theorem 3.19], which, using this information on the maximum energy barrier, yield sharp bounds in probability for the hitting times $\tau^{\mathbf{s}}_{\mathbf{s}'}$ and $\tau^{\mathbf{s}}_{\mathcal{X}^s \setminus \{\mathbf{s}\}}$ and the convergence of their scaled first moments. The technical assumption under which these two results hold is implied by (11) and [56, Proposition 3.18].

Let us then prove inequality (11). If $\sigma \in \mathcal{X}^s \setminus \{\mathbf{s}'\}$, then the inequality follows from Theorem 2.1(i). In the other case, when $\sigma \notin \mathcal{X}^s$, Theorem 2.1(ii) guarantees that there exists $\mathbf{s}^* \in \mathcal{X}^s$ such that $\Phi(\sigma, \mathbf{s}^*) - H(\sigma) < \Gamma(\Lambda)$, which means that there exists a path $\omega^* : \sigma \to \mathbf{s}^*$ with $\Phi_{\omega^*} - H(\sigma) < \Gamma(\Lambda)$. If $\mathbf{s}^* = \mathbf{s}'$, then the claim in (11) is proved. Otherwise, we can concatenate such path ω^* and another path, say $\omega^{(2)} : \mathbf{s}^* \to \mathbf{s}'$, constructed as in Proposition 2.4 to obtain a path $\omega : \sigma \to \mathbf{s}'$ satisfying $\Phi_\omega - H(\sigma) \leq \Gamma(\Lambda)$, proving that inequality (11) holds also in this case.

(iii) Since the statement of Theorem 2.1(i) holds for any pair of stable configurations, it follows that $\Phi(\mathbf{s}, \mathcal{X}^s \setminus \{\mathbf{s}\}) - H(\mathbf{s}) = \Gamma(\Lambda)$ for every $\mathbf{s} \in \mathcal{X}^s$. Combining this identity with Theorem 2.1(ii) readily implies that

$$\forall \mathbf{s} \in \mathcal{X}^s \qquad \max_{\sigma \in \mathcal{X} \setminus \mathcal{X}^s} \Phi(\sigma, \mathcal{X}^s) - H(\sigma) < \Phi(\mathbf{s}, \mathcal{X}^s \setminus \{\mathbf{s}\}) - H(\mathbf{s}). \tag{12}$$

This inequality means that the energy barrier separating $\mathbf{s}$ from the target set $\mathcal{X}^s \setminus \{\mathbf{s}\}$ is strictly larger than any other energy barrier in the energy landscape. The technical condition in [56, Proposition 3.20] then holds and we can apply [56, Theorem 3.19] to get the asymptotic exponentiality of $\tau^{\mathbf{s}}_{\mathcal{X}^s \setminus \{\mathbf{s}\}} / \mathbb{E}\tau^{\mathbf{s}}_{\mathcal{X}^s \setminus \{\mathbf{s}\}}$ and conclude the proof.

(iv) If $q = 2$, statements (iii) and (iv) coincide and there is nothing to prove. If $q > 2$, although statement (iv) look very similar to (iii) and (iv), its proof does not immediately follow from the structural properties of the energy landscape. Indeed in this case the subset $\mathcal{X}^s \setminus \{\mathbf{s}, \mathbf{s}'\}$ is not empty and there exists at least a third stable configuration $\eta \in \mathcal{X}^s \setminus \{\mathbf{s}, \mathbf{s}'\}$ such that

$\Phi(\mathbf{s},\mathbf{s}') - H(\mathbf{s}) \not> \Phi(\eta,\mathbf{s}') - H(\eta)$ (both the left-hand and right-hand sides are equal to $\Gamma(\Lambda)$ by Theorem 2.1). Thus, a condition analogous to (12) does not hold in this case and thus we cannot directly apply the model-independent results in [56] as we did for statement (iii). This technical detail reflects the fact that the energy landscape $\mathcal{X} \setminus \{\mathbf{s},\mathbf{s}'\}$ has at least one stable configuration different from the starting and target ones; the dynamics may be trapped in the "valley" of any such stable configuration and thus these visits must be considered to determine the asymptotic distribution of $\tau^{\mathbf{s}}_{\mathbf{s}'}/\mathbb{E}\tau^{\mathbf{s}}_{\mathbf{s}'}$. The proof of the asymptotic exponentiality of $\tau^{\mathbf{s}}_{\mathbf{s}'}$ is thus obtained leveraging the already proved statement (iii) in combination with a stochastic representation of the tunneling time $\tau^{\mathbf{s}}_{\mathbf{s}'}$ that exploits the intrinsic symmetries of the energy landscape $(\mathcal{X}, H, Q)$ of the q-state Potts model on Λ. For any $k, l \in \{1,\dots,q\}$, with $k \neq l$, define the automorphism $\Psi_{k,l} : \mathcal{X} \to \mathcal{X}$ as

$$[\Psi_{k,l}(\sigma)](v) = \begin{cases} \sigma(v) & \text{if } \sigma(v) \neq k, l, \\ k & \text{if } \sigma(v) = l, \\ l & \text{if } \sigma(v) = k. \end{cases}$$

In other words, the configuration $\Psi_{k,l}(\sigma)$ is obtained from σ by interchanging every spin with value k with a spin with value l and vice-versa, while leaving all the other $q-2$ spin values unchanged. Exploiting the family of automorphisms $\{\Psi_{k,l}\}_{k,l=1,\dots,q}$ and, arguing similarly to [67, Proposition 2] and [66, Proposition 4.3], we can construct a coupling between different copies of the Markov chain $\{X^{\beta}_t\}_{t\in\mathbb{N}}$ and show that for any $\mathbf{s} \in \mathcal{X}^s$ and at any temperature $\beta > 0$ the following properties hold:

(a) The random variable $X_{\tau^{\mathbf{s}}_{\mathcal{X}^s\setminus\{\mathbf{s}\}}}$ has a uniform distribution over $\mathcal{X}^s \setminus \{\mathbf{s}\}$;
(b) The distribution of the random variable $\tau^{\mathbf{s}}_{\mathcal{X}^s\setminus\{\mathbf{s}\}}$ does not depend on $\mathbf{s}$;
(c) The random variables $\tau^{\mathbf{s}}_{\mathcal{X}^s\setminus\{\mathbf{s}\}}$ and $X_{\tau^{\mathbf{s}}_{\mathcal{X}^s\setminus\{\mathbf{s}\}}}$ are independent.

Let N_q be the random variable that counts the number of non-consecutive visits to stable configurations in $\mathcal{X}^s \setminus \{\mathbf{s}'\}$ until the configuration $\mathbf{s}'$ is hit (counting the initial configuration $\mathbf{s}$ where the Markov chain starts at time $t = 0$ as first visit). Non-consecutive visits means that we count as actual visit to a stable configuration only the first one after the last visit to a different stable configuration. Property (b) implies that the random time between these non-consecutive visits does not depend on the last visited stable configuration. In view of property (a), the random variable N_q is geometrically distributed with success probability equal to $(q-1)^{-1}$, i.e.,

$$\mathbb{P}(N_q = m) = \left(1 - \frac{1}{q-1}\right)^{m-1} \cdot \frac{1}{q-1}, \quad m \geq 1. \tag{13}$$

In particular, N_q depends only on q and *not* on the inverse temperature β. The amount of time $\tau^{\mathbf{s}}_{\mathcal{X}^s\setminus\{\mathbf{s}\}}$ it takes the Metropolis Markov chain started in $\mathbf{s} \in \mathcal{X}^s$ to hit any stable configuration in $\mathcal{X}^s \setminus \{\mathbf{s}\}$ does not depend on $\mathbf{s}$, by virtue of property (b). In view of these considerations and using the independence property (c), we derive for every $\mathbf{s},\mathbf{s}' \in \mathcal{X}^s$, $\mathbf{s} \neq \mathbf{s}'$ the following stochastic representation of the tunneling time $\tau^{\mathbf{s}}_{\mathbf{s}'}$:

$$\tau^{\mathbf{s}}_{\mathbf{s}'} \stackrel{d}{=} \sum_{i=1}^{N_q} \tau^{(i)}, \tag{14}$$

where $\{\tau^{(i)}\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables distributed as $\tau^{\mathbf{s}}_{\mathcal{X}^s\setminus\{\mathbf{s}\}}$ and N_q is an independent geometric random variable as defined in (13). In particular, since both random variables N_q and $\tau^{\mathbf{s}}_{\mathcal{X}^s\setminus\{\mathbf{s}\}} \stackrel{d}{=} \tau^{(i)}$ have finite expectation and $\mathbb{E}N_q = q-1$, it immediately follows from Wald's identity that $\mathbb{E}\tau^{\mathbf{s}}_{\mathbf{s}'} = (q-1) \cdot \mathbb{E}\tau^{\mathbf{s}}_{\mathcal{X}^s \setminus \{\mathbf{s}\}} = (q-1) \cdot \mathbb{E}\tau^{(i)}$. Thus, we can rewrite (14) as $\tau^{\mathbf{s}}_{\mathbf{s}'}/\mathbb{E}\tau^{\mathbf{s}}_{\mathbf{s}'} \overset{d}{=} (\mathbb{E}N_q)^{-1} \sum_{i=1}^{N_q} \tau^{(i)}/\mathbb{E}\tau^{(i)}$. Statement (iii) gives that $\tau^{(i)}/\mathbb{E}\tau^{(i)} \xrightarrow{d} \mathrm{Exp}(1)$ for every $i \in \mathbb{N}$ as $\beta \to \infty$ and thus $\tau^{\mathbf{s}}_{\mathbf{s}'}/\mathbb{E}\tau^{\mathbf{s}}_{\mathbf{s}'}$ is asymptotically distributed as geometric sum of i.i.d. unit-mean exponential random variables, which is also exponentially distributed.

(v) By combining Theorem 2.1(i) and (ii), it is easy to check that $\max_{\sigma \neq \mathbf{s}} \Phi(\sigma, \mathbf{s}) - H(\sigma) = \Gamma(\Lambda)$ for every $\mathbf{s} \in \mathcal{X}^s$, and the statements for both mixing time and spectral gap then follow from [56, Proposition 3.24]. □

Acknowledgment

The research of A.Z. is currently supported by NWO, Netherlands grant 680.50.1529.

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